

A quark action for very coarse lattices

Mark Alford

School of Natural Sciences
Institute for Advanced Study
Princeton, NJ 08540

Timothy R. Klassen

SCRI
Florida State University
Tallahassee, FL 32306

G. Peter Lepage

Lab of Nuclear Studies
Cornell University
Ithaca, NY 14853

February 1, 2008

Abstract

We investigate a tree-level $\mathcal{O}(a^3)$ -accurate action, D234c, on coarse lattices. For the improvement terms we use tadpole-improved coefficients, with the tadpole contribution measured by the mean link in Landau gauge.

We measure the hadron spectrum for quark masses near that of the strange quark. We find that D234c shows much better rotational invariance than the Sheikholeslami-Wohlert action, and that mean-link tadpole improvement leads to smaller finite-lattice-spacing errors than plaquette tadpole improvement. We obtain accurate ratios of lattice spacings using a convenient “Galilean quarkonium” method.

We explore the effects of possible $\mathcal{O}(\alpha_s)$ changes to the improvement coefficients, and find that the two leading coefficients can be independently tuned: hadron masses are most sensitive to the clover coefficient C_F , while hadron dispersion relations are most sensitive to the third derivative coefficient C_3 . Preliminary non-perturbative tuning of these coefficients yields values that are consistent with the expected size of perturbative corrections.

1 Introduction

Lattice QCD remains the only complete implementation of nonperturbative QCD and so is essential for low-energy QCD phenomenology. However, simulations of lattice QCD rely upon brute force Monte Carlo evaluations of the QCD path integral, and are very costly. In recent years it has been demonstrated that this cost is dramatically reduced by using coarse lattices, with lattice spacings as large as $a = 0.4$ fm, together with more accurate discretizations of QCD. While highly corrected discretizations of gluon and heavy-quark actions are now commonplace, less progress has been made with the much harder problem of constructing highly improved light-quark actions. The best light-quark actions in widespread use have finite- a errors proportional to a^2 , which are large compared with the $a^4, \alpha_s^2 a^2$ errors for improved gluon actions. The problem is compounded by the fact that the effective lattice spacing for light quarks is $2a$ rather than a , because the light-quark action, unlike the others, involves first-order derivatives. There are a number of problems, like relativistic heavy-quark physics and high-momentum form factors, where $\mathcal{O}(a^2)$ improvement is crucial (in particular when used in conjunction with anisotropic lattices) for accurate results without more or less uncontrolled extrapolations over large mass and/or momentum regions. In this paper we take a step towards remedying this situation by presenting new results obtained using a highly corrected lattice action for light-quarks.

The finite- a errors can be removed, order-by-order in a , from a lattice lagrangian by adding correction terms:

$$\mathcal{L} = \mathcal{L}_0 + \sum_i a^{n_i} c_i \mathcal{L}_i. \quad (1.1)$$

In principle, the coefficients c_i of these correction terms can be computed using (weak-coupling) perturbation theory, but in lattice QCD there has been a long-term reluctance to rely on perturbation theory for any of the ingredients in QCD simulations. For most of the past twenty years this has meant that no correction terms were included in the action, which then has only one parameter, the bare quark mass; the mass is tuned nonperturbatively to give correct hadron masses. Recently a practical technique has been developed for nonperturbatively computing the coefficient of the $\mathcal{O}(a)$ correction[1]. The $\mathcal{O}(a)$ -accurate quark action, originally discussed in [2], has led to substantial improvements over past work, but it is still of limited value for lattice spacings larger than 0.1–0.2 fm.

With only a few exceptions, it is very difficult to compute the coefficients for $\mathcal{O}(a^2)$ and higher corrections nonperturbatively. Thus a perturbative determination is the only practical alternative that permits further improvement. Given the advantages of very coarse lattices, we feel it is too restrictive to abandon perturbation theory completely. This is particularly the case since we now know

that perturbation theory is generally quite reliable, provided one uses tadpole-improved lattice operators [3, 4]. In particular, perturbation theory correctly *predicted* the relatively large renormalization of the $\mathcal{O}(a)$ correction to the quark action several years before it was confirmed in nonperturbative studies.

The coupling constant, $\alpha_s(\pi/a)$, is larger on coarser lattices, and therefore perturbation theory is less convergent. This makes perturbation theory less practical for calculating such things as the overall renormalization factors relating lattice currents to continuum currents. The correction terms in the quark action, however, are suppressed by explicit powers of the lattice spacing. Consequently they require less high precision, and even low-order perturbation theory may suffice for results accurate at the few percent level.

In this paper we derive a tadpole-improved $\mathcal{O}(a^3)$ -accurate quark action, “D234c”. We compare its predictions with the those of the standard Sheikholeslami-Wohlert (SW) $\mathcal{O}(a)$ -accurate action, and also with the original Wilson (W) action. To study finite-lattice-spacing errors it is not necessary to take the chiral limit, so we restrict our study to quark masses near the strange quark’s mass. Since finite- a errors typically grow with quark mass, our results should improve for u and d quarks.

The important points in our analysis are:

- We use the mean link in Landau gauge rather than the traditional plaquette prescription for calculating our tadpole improvement factor u_0 . Our reasons are: (1) it gives a more rotationally invariant static potential [5]; (2) it has been shown in NRQCD that it leads to smaller scaling errors in the charmonium hyperfine splitting [6]; (3) for Wilson glue, it gives a clover coefficient that agrees more closely with the non-perturbatively determined value [4]. These studies suggest that the mean-link tadpole prescription has smaller quantum corrections than the plaquette prescription. Of course, once higher order perturbative corrections are included the two prescriptions will come into agreement [4].
- After tadpole improvement, the improvement coefficients are expected to have quantum corrections of order α_s , which is ~ 0.4 on our coarsest lattice. In Monte-Carlo simulations, we systematically study the effects of corrections of this size, and find that the clover coefficient C_F is the only one whose quantum corrections will affect hadron masses significantly, and the third derivative coefficient C_3 is the only one that affects hadron dispersion relations significantly.
- We perform various non-perturbative tests of the coefficients of the improvement terms. We measure the hadron dispersion relation (“speed of

light”) to check the $a^2\Delta^{(3)}$ term; vector meson (ϕ) scaling as a check on the relative weight of the clover and Wilson terms; r -dependence as an additional check on the relative weight of the Wilson and clover terms and the effects of ghost branches in the quark dispersion relation.

- We perform a rough non-perturbative tuning of the two leading coefficients of the $D234$ action, and discuss the comparison with perturbative expectations.
- We set our overall scale from the charmonium $P - S$ splitting. However for comparisons of scaling it is ratios of lattice spacings that are important, and we introduce a simple method for determining these more accurately, and with less vulnerability to systematic errors. It consists of measuring $P - S$ in a fictitious heavy quark “Galilean quarkonium” state, i.e. using an NRQCD heavy quark action in which relativistic corrections are not included. Varying the quark mass changes the size of the state, and thereby tests for the presence of finite- a errors.

We have previously studied a plaquette-tadpole-improved $\mathcal{O}(a^2)$ -accurate action on isotropic lattices [7], and plaquette-tadpole-improved $\mathcal{O}(a^3)$ -accurate actions on anisotropic lattices [8, 9], and found good dispersion relations and scaling of mass ratios. In this paper we find that mean-link tadpole improved $D234c$ has the same benefits, plus much smaller finite- a errors in hadron masses. This is as expected, because mean-link tadpole improvement gives a larger clover coefficient.

2 $D234c$ Quark action

Following [9], we construct a quark action that is continuum-like (at tree level) through $\mathcal{O}(a^3)$. We start with the continuum quark action:

$$S = \bar{\psi}_c M_c \psi_c, \quad M_c = \not{D} + m_c \quad (2.1)$$

If we discretize this directly, our quark dispersion relation will contain unwanted doublers at the edges of the Brillouin zone. To avoid this, we perform a field redefinition, parameterized by r , before discretizing:

$$\begin{aligned} \psi_c &= \Omega \psi \\ \bar{\psi}_c &= \bar{\psi} \bar{\Omega} \end{aligned} \quad (2.2)$$

where

$$\Omega^2 = \bar{\Omega}^2 = 1 - \frac{1}{2} r a_t (\not{D} - m_c). \quad (2.3)$$

Now $S = \bar{\psi} M \psi$, where the transformed continuum quark operator is

$$M = \bar{\Omega} M_c \Omega = \not{D} + m_c - \frac{1}{2} r a (\not{D}^2 - m_c^2). \quad (2.4)$$

We use $\not{D}^2 = \sum_{\mu} D_{\mu}^2 - \frac{1}{2} \sigma \cdot F$, and discretize, allowing errors of order a^4 , to obtain the lattice D234c quark action:

$$\begin{aligned} M_{\text{D234c}} = & m_c \left(1 + \frac{1}{2} r a m_c\right) + \sum_{\mu} \left\{ \gamma_{\mu} \Delta_{\mu}^{(1)} - \frac{C_3}{6} a^2 \gamma_{\mu} \Delta_{\mu}^{(3)} \right. \\ & \left. + r \left[-\frac{1}{2} a \Delta_{\mu}^{(2)} - \frac{C_F}{4} a \sum_{\nu} \sigma_{\mu\nu} F_{\mu\nu} + \frac{C_4}{24} a^3 \Delta_{\mu}^{(4)} \right] \right\}. \end{aligned} \quad (2.5)$$

$\Delta_{\mu}^{(n)}$ is the most local centered lattice discretization of the gauge-covariant n 'th derivative [9, 10]; $\Delta^{(3)} = \Delta^{(1)} \Delta^{(2)} = \Delta^{(2)} \Delta^{(1)}$, and $\Delta^{(4)} = \Delta^{(2)} \Delta^{(2)}$. The field strength consists of the standard clover term $F_{\mu\nu}^{(cl)}$, and a relative $\mathcal{O}(a^2)$ correction [9] with coefficient C_{2F} :

$$F_{\mu\nu}(x) \equiv F_{\mu\nu}^{(cl)}(x) - a^2 \frac{C_{2F}}{6} (\Delta_{\mu}^{(2)} + \Delta_{\nu}^{(2)}) F_{\mu\nu}^{(cl)}(x) \quad (2.6)$$

At tadpole-improved tree level, all links are divided by the Landau gauge mean link u_0 , and $C_F = C_3 = C_4 = C_{2F} = 1$. We will explore the effects of deviations from these values.

The terms proportional to r remove the doublers from the quark dispersion relations, so that for generic values of $r \sim 1$ this is a doubler-free tree-level $\mathcal{O}(a^3)$ -accurate quark action. The derivation can be straightforwardly generalized to anisotropic lattices [9, 10].

For $r = 1$ there are three fairly high ghost branches in the free quark dispersion relation (Figure 1). To investigate the effect of redundant terms we will also study $r = 2/3$, for which one of the ghost branches moves down so that $E(0) \approx 1.0$.

Note that the two leading terms both violate symmetries that will be restored in the continuum limit, and hence can be non-perturbatively tuned. The clover term violates chiral symmetry, and so can be tuned by imposing PCAC [1]. The $\Delta^{(3)}$ term is the *only* rotational symmetry-violating $\mathcal{O}(a^2)$ term, and so its coefficient C_3 can be nonperturbatively tuned by imposing rotational invariance.

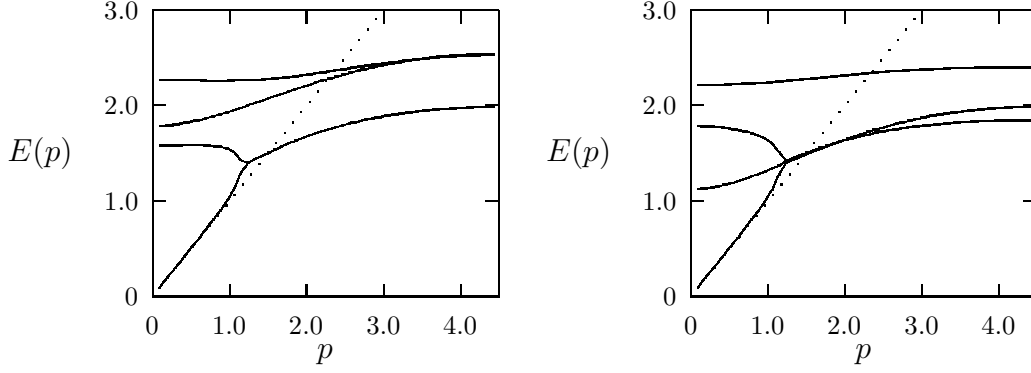


Figure 1: Massless dispersion relation $\text{Re}[aE(a|\mathbf{p}|)]$ for D234c($r = 1$) and D234c($r = 2/3$) quarks on an isotropic lattice. Note the continuum-like behavior at low momentum, and the presence of three ghost branches, one of which drops dramatically as r changes from 1 to $2/3$.

3 Gluon action and lattice spacing determination

We use a tree-level tadpole-improved plaquette and 2×1 rectangle glue action [11, 12, 13],

$$\begin{aligned}
S &= -\beta \sum_{x,\mu} \left\{ \frac{5}{3} \frac{P_{\mu\nu}(x)}{u_0^4} - \frac{1}{12} \frac{R_{\mu\nu}(x)}{u_0^6} - \frac{1}{12} \frac{R_{\nu\mu}(x)}{u_0^6} \right\}, \\
P_{\mu\nu} &= \frac{1}{3} \text{ReTr} W(\mu + \nu - \mu - \nu) \quad (\text{plaquette}), \\
R_{\mu\nu} &= \frac{1}{3} \text{ReTr} W(2\mu + \nu - 2\mu - \nu) \quad (\text{rectangle}),
\end{aligned} \tag{3.1}$$

where a Wilson loop $W(\rho - \sigma \cdots)$ goes one link in the ρ direction, one link in the negative σ direction, etc.

This definition of β is different from that of Ref. [13], where a factor of $5/3u_0^4$ was absorbed into β . We prefer the notation here because $\beta = 6/g^2$ as in the original Wilson action. Furthermore the coupling

$$\alpha_s \equiv \frac{3}{2\pi\beta} \tag{3.2}$$

is now tadpole-improved and therefore roughly equal to continuum couplings like $\alpha_V(\pi/a_s)$.

The tadpole improvement factor u_0 is the mean of the link operator in Landau gauge. At both our lattice spacings we found that an 8^4 lattice was large enough for finite volume effects in u_0 to be of order 0.1%. To fix to Landau gauge we

maximize [4]

$$\sum_{x,\mu} \frac{1}{u_0 a^2} \text{ReTr} \left\{ U_\mu(x) - \frac{1}{16u_0} U_\mu(x) U_\mu(x + \hat{\mu}) \right\}. \quad (3.3)$$

We generated gluon configurations at two lattice spacings, 0.40 fm and 0.25 fm (see Table 1). The lattice spacing was determined in two ways.

Firstly, we performed NRQCD simulations of charmonium, using the experimental value of 458 MeV for the spin-averaged $P - S$ splitting; the results are given in Table 1 column 6 (details in Table 7). Note that the errors quoted are statistical, and do not reflect systematic uncertainties such as quenching, finite- a errors, or higher-order relativistic effects neglected in our NRQCD simulation. This lattice spacing determination is therefore not suitable for precise comparisons with data from other groups.

Secondly, a more accurate determination of *ratios* of lattice spacings is possible, since there is no need to simulate a known physical state. At each lattice spacing we can measure the mass of a fictitious state, whose properties are chosen for convenience. We chose “Galilean quarkonium,” a bound state of a quark and antiquark in a non-relativistic world. We simulated this state using NRQCD with no relativistic corrections. By making the Galilean quarkonium lighter (and hence bigger) than charmonium we reduce the finite- a errors. In fact, we studied a range of quark masses down to about half the charm quark mass and found that lattice spacing ratios were all consistent with each other, within errors. For details see Appendix A and Table 6. Since our main goal is to compare our mean-link-tadpole-improved results with SCRI’s plaquette-tadpole-improved results, we used Galilean quarkonium to calculate the ratios of our lattice spacings to the SCRI $\beta = 7.4$ lattice spacing. The results are in Table 6.

For convenience we want to give our results an absolute energy scale, so we take the SCRI $\beta = 7.4$ lattice spacing to be $a^{-1} = 812$ MeV, which corresponds to $\sqrt{\sigma} = 468$ MeV for their string tension. This gives the final column of Table 1, which is consistent with our charmonium measurements. Note that the error bars reflect the uncertainty in the ratio to SCRI’s lattice spacings, which is the relevant quantity for scaling comparisons. It does not reflect the overall uncertainty in the scale, which was introduced purely for convenience.

For hadron spectrum measurements we used lattices of the same physical size (2 fm) at both lattice spacings. We also performed a set of measurements investigating the volume dependence of hadron masses (see appendix, Table 8). We see that the 1.6 fm lattice agrees with the 1.75 fm and 2 fm lattices within statistical errors.

β	α_s	u_0	u_p	approx a	charmonium a^{-1}	a^{-1} rel. to SCRI
1.157	0.413	0.738	0.8196	0.40 fm	495(4) MeV	497(3) MeV
1.719	0.278	0.797	0.8576	0.25 fm	790(10) MeV	785(6) MeV
7.4 (SCRI)				0.24 fm	840(20) MeV	812(def) MeV

Table 1: Parameters of glue for our two lattice spacings, and one of SCRI’s. Mean Landau-gauge link (u_0) was used in our tadpole improvement (TI); plaquette TI factor $u_p = (\text{Plaq})^{1/4}$ is given for comparison. The a^{-1} relative to SCRI is obtained by the Galilean quarkonium method.

4 Results

We subjected the D234c action to a series of tests to determine its viability at large lattice spacings. We examined the scaling of the vector meson mass and of baryon masses, and we measured hadronic dispersion relations. We also measured the sensitivity of these physical quantities to changes in the coefficients in the action.

4.1 Hadron masses

In Figure 2 we show how the vector meson mass varies with lattice spacing when the ratio of the pseudoscalar to vector meson masses is $P/V = 0.7$. (Full data is in Appendix B , along with data for $P/V = 0.76$, corresponding to a slightly larger quark mass; see tables 10, 9). We present data obtained using the tree-level D234c, with Landau-link tadpole improvement, at lattice spacings a of 0.25 fm and 0.4 fm. These values are compared with results from SCRI obtained using the Wilson and SW actions [14], which can be extrapolated to give $a=0$ results, as indicated.

We also examined baryon masses; ratios of these to the vector mass are shown in Fig. 3. Our measured values at $a = 0.25$ fm are within 2σ of the quadratic extrapolation of the SCRI values to the continuum.

a	mean-link D234c	mean-link SW	plaquette SW	plaquette Wilson
0.4	0.969(9)	0.933(10)		
0.25	1.027(10)	1.041(15)		
0			1.035(5)	1.034(37)

Table 2: Phi (V) masses in GeV at $P/V = 0.70$, from tables 1 and 10, using a quadratic fit to SCRI’s SW data to extrapolate it to $a = 0$. For the D234c action we see a 2(1)% scaling error at $a = 0.25$ fm, and 7(1)% scaling error at $a = 0.4$ fm.

In table 2 we give the vector mass at our two lattice spacings, along with a naive

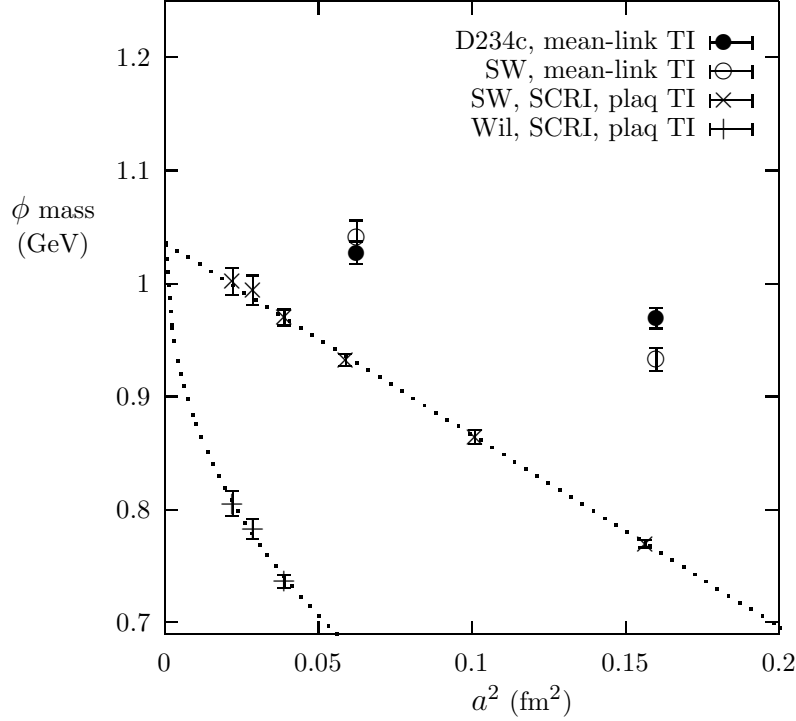


Figure 2: Mass of ϕ meson as a function of lattice spacing, $P/V = 0.70$. Mean link TI (\bullet D234, \circ SW) clearly scales better than plaquette TI (\times SW, $+$ Wilson, data kindly supplied by SCRI)

continuum extrapolation of SCRI's data. (For details of the scale determination see Appendix A). The errors include the relative error between our mass scale and SCRI's, but not the uncertainty in the absolute scale, which may be affected by quenching effects.

Using a quadratic fit to SCRI's SW data to extrapolate it to $a = 0$, the D234c action shows a 2(1)% finite- a error at $a = 0.25$ fm, and 7(1)% finite- a error at $a = 0.4$ fm. (The quadratic fit may be too naive: the true continuum value could differ by a few percent, however this will not substantially affect our conclusions below.) These finite- a errors are due to radiative corrections to the tree-level coupling constants, and higher-order interactions not included in our action. We measured the sensitivity of the hadron masses to radiative corrections by varying each of the tree-level coupling constants. The fractional change caused by multiplying each coupling constant in turn by $1 + \alpha_s$ is shown in Table 3. The only coupling for which radiative corrections are important is the clover coupling, C_F [15]. A perturbative analysis of C_F through $\mathcal{O}(\alpha_s)$ will soon be completed [16]. Alternatively, the $\mathcal{O}(\alpha_s)$ coefficient could be determined

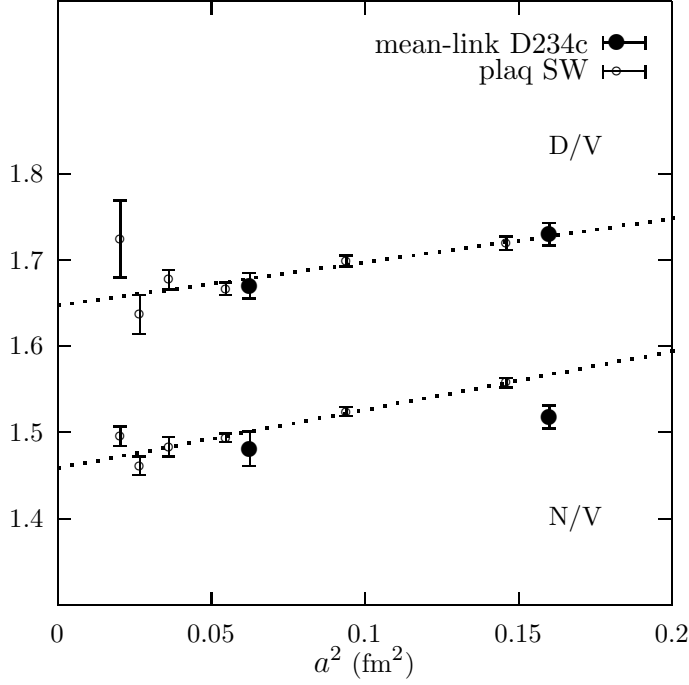


Figure 3: D/V and N/V for D234c and plaq TI SW (SW data kindly supplied by SCRI.)

by making the vector meson mass at our smallest lattice spacing agree with the continuum. For example, using our continuum extrapolation of SCRI's data we find that taking

$$C_F = 1 + 0.2\alpha_s, \quad (4.1)$$

where α_s is the bare coupling (3.2), reduces the V mass error at 0.25 fm from 2(1)% to 0(1)%, and at 0.4 fm from 7(1)% to 0(1)%. This suggests that perturbative corrections to C_F are relatively small after tadpole improvement. The necessary corrections to C_F are still perturbative if the true continuum value differs by a few percent from the naive extrapolation of the SCRI data.

We also include SW results in Figure 2 using the Landau-link tadpole-improved tree-level value for the clover coefficient, C_F . Comparing these with SCRI's SW results, for which the plaquette was used to determine u_0 , we see that the Landau-link results show much smaller finite- a errors. This result was anticipated based on work at smaller lattice spacings using SW quarks and the (unimproved) Wilson action for the gluons [4].

coeff		C_F	C_3	C_4	C_{2F}
a (fm)	0.25	0.4	0.4	0.4	0.4
P	11(1)%	35(1)%	3.8(6)%	-1.0(2)%	-2.9(9)%
V	10(1)%	35(1)%	3.7(4)%	-1.0(2)%	-2.9(5)%
N	6(2)%	26(2)%	3.4(7)%	-0.9(2)%	-1.7(7)%
D	8(1)%	25(2)%	3.8(9)%	-1.1(3)%	-2.3(4)%

Table 3: Percentage change in hadron masses when individual coefficients in the mean-link tree-level TI D234c quark action (Eq. 2.5) are multiplied by $1 + \alpha_s$. All are at $P/V \approx 0.7$, except the $C_F, a = 0.4$ is at $P/V = 0.76$.

4.2 Hadron dispersion relations

The finite- a errors in the ϕ mass appear to be almost as small for the SW action as they are for D234c, but this is deceptive. To see why, we consider the quantity

$$c^2(\mathbf{p}) = \frac{E^2(\mathbf{p}) - E^2(0)}{\mathbf{p}^2} \quad (4.2)$$

for different hadrons and three momenta \mathbf{p} , where $E(\mathbf{p})$ is the hadron’s energy. In the continuum limit, $c^2(\mathbf{p}) = 1$ for all \mathbf{p} . This quantity is particularly sensitive to the C_3 term in the D234c action since this term is not rotationally invariant; it cancels the leading (non-rotationally invariant) error in the SW action. Our results for c^2 , for both pseudoscalar and vector mesons, are shown in Figure 4. At 0.4 fm, D234c is dramatically superior: it deviates from $c^2 = 1$ by only 3–5% at zero momentum, and by less than 10% even at momenta of order $1.5/a$, while SW gives results that deviate by 40–60% or more for all momenta, including zero. As expected both formalisms improve at 0.25 fm, although D234c is still clearly superior.

The c^2 results have practical implications. For example, there are two different definitions of a hadron’s mass in lattice simulations. One is the “static mass,”

$$m_{\text{static}} \equiv E(p=0), \quad (4.3)$$

and the other the “kinetic mass,”

$$m_{\text{kinetic}} \equiv \lim_{p \rightarrow 0} \frac{p}{dE/dp} = E(0)/c^2(0). \quad (4.4)$$

In D234c these two definitions agree to within 3–5% at 0.4 fm. In SW they differ by 40–60%, making it impossible to say what the “true” mass is for this formalism

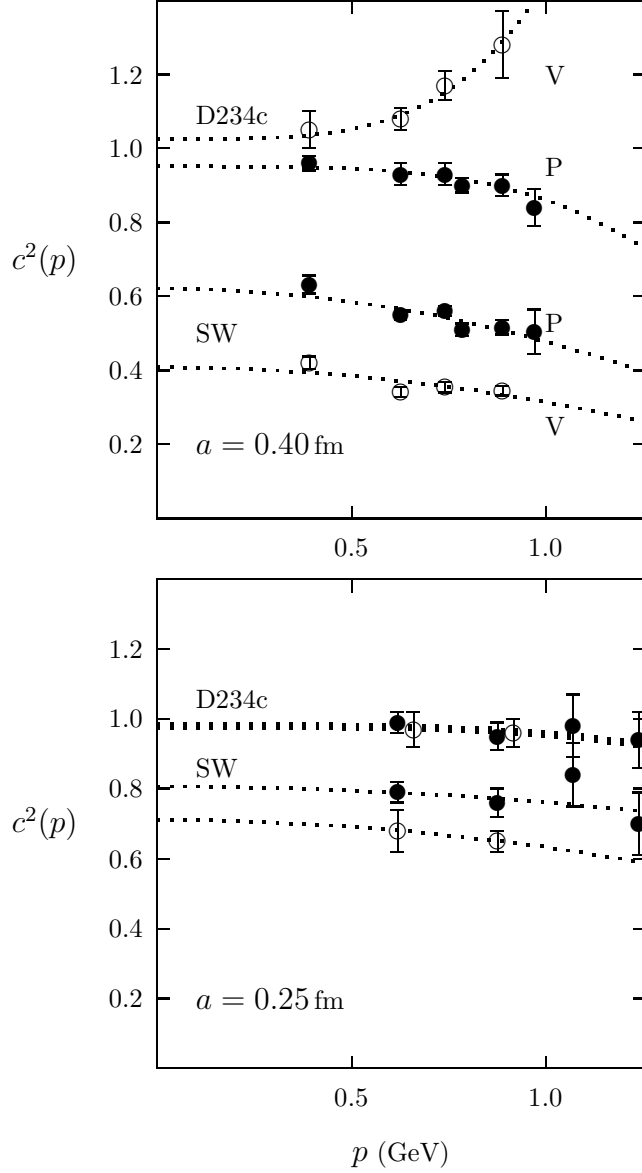


Figure 4: Speed of light squared using the D234c and SW actions, for the pseudoscalar (●) and vector (○) mesons, at $P/V = 0.76$, on a $5^2 \times 8 \times 18$ lattice at $a = 0.4$ fm (see Table 12), and on a $8^3 \times 24$ lattice at $a = 0.25$ fm (see Table 13). [D234c V points are offset for clarity.] D234c shows much better rotational invariance; both discretizations improve at the smaller lattice spacing, where the pseudoscalar and vector results for D234c are indistinguishable with our statistics.

at 0.4 fm. The kinetic and static masses in either formalism must be equal for zero-mass mesons, because of the axis-interchange symmetry of the actions. The deviations seen here are because the strange quark is relatively massive at our lattice spacings: the ϕ mass, for example, is $2.1/a$ with D234c at 0.4 fm. These results illustrate that D234c is far more accurate than SW for hadrons with large masses in lattice units. More generally, D234c is far more accurate for hadrons with large energies and/or large momenta.

The C_3 term is the only a^2 correction that breaks rotational invariance. Thus we can use c^2 to tune C_3 nonperturbatively. In our 0.4 fm simulation, we tuned C_3 to make the dispersion relation for the lightest meson, the pseudoscalar, perfect at low momentum: when

$$C_3 = 1.2 \tag{4.5}$$

$$\approx 1 + 0.5\alpha_s \tag{4.6}$$

we obtain the results shown in the top part of Figure 5. The c^2 for the pseudoscalar is now within $\pm 2\%$ of $c^2 = 1$ at least out to momenta of order $1.5/a$.

It is noticeable in figure 5 that tuning the pseudoscalar c^2 has worsened the dispersion relation for the vector meson ($c_V^2(p=0) = 1.21(5)$ from a quartic fit). However, the vector is 25% heavier than the pseudoscalar and so should have substantially larger $(am_V)^n$ errors. This is confirmed by a rerun at lower quark mass, $P/V = 0.6$, where the vector is lighter (bottom part of Figure 5). There we see that choosing $C_3 = 1.2$ works for both pseudoscalar and vector: the vector now has $c^2(p=0) = 1.04(4)$. In table 11 we see that, to within statistical errors, the pseudoscalar meson dispersion relation is insensitive to C_F , confirming that C_3 can be tuned independently of C_F . (We ignore the vector meson because, as noted above, at $a = 0.4$ fm and $P/V = 0.7$ it is too heavy for its dispersion relation to be a reliable indicator of the Lorentz-violating errors.)

From these results it is clear that for the D234c action, as for SW, the hadron dispersion relation is more sensitive to finite- a errors than the hadron mass. At $P/V = 0.7$ on an isotropic 0.4 fm lattice we get a perfectly sensible vector mass (Fig. 2), but the dispersion relation is beginning to break down due to $\mathcal{O}((am)^2)$ errors. An anisotropic lattice would extend D234c's range to much higher quark masses at the same spatial lattice spacing, because the errors due to the quark mass are of order $(a_tm)^2$.

4.3 Redundant terms

As a final test of the D234c action, we varied the parameter r . The r -dependent terms in the D234c action are, by design, approximately redundant, and therefore varying r should have little effect on the spectrum. Furthermore the extent to

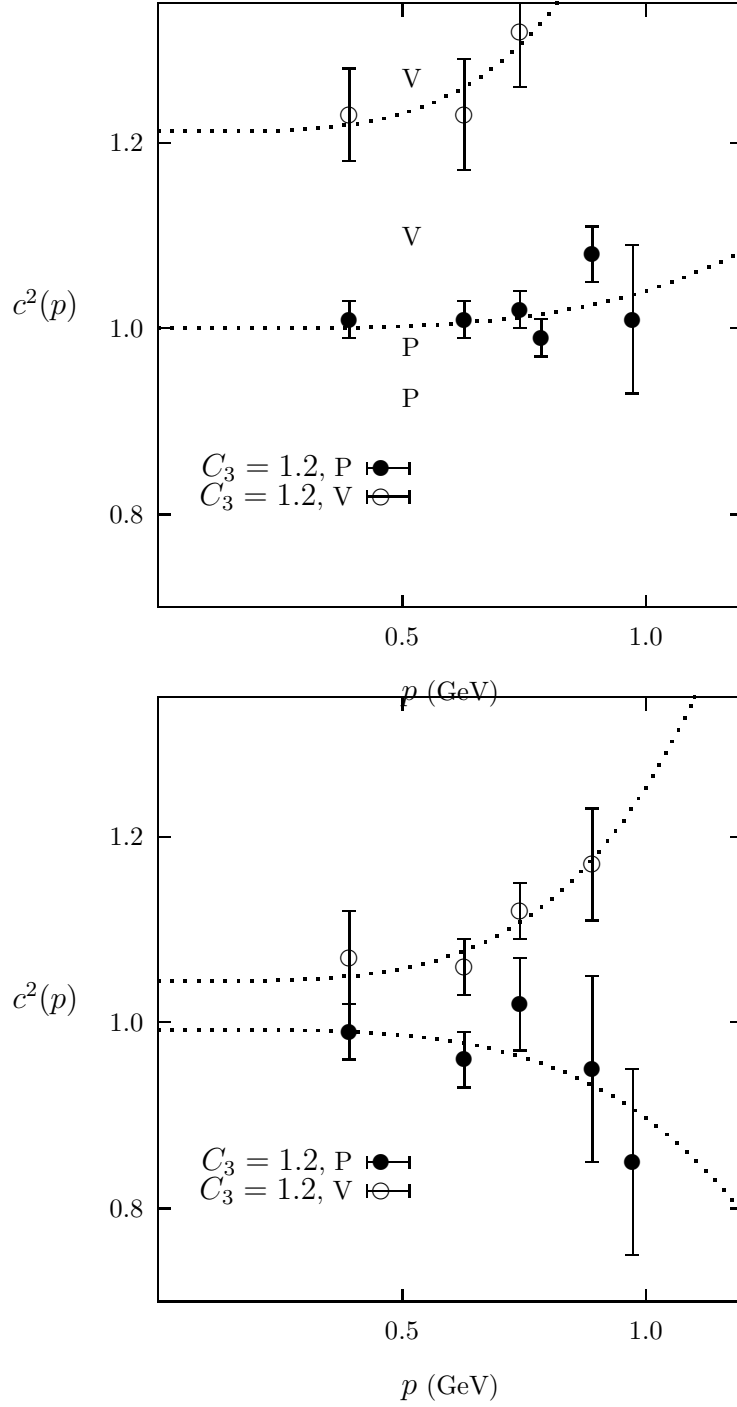


Figure 5: Speed of light squared using the D234c action with coupling $C_3 = 1.2$ for pseudoscalar (P) and vector (V) mesons, at $P/V = 0.76$ (upper figure) and $P/V = 0.6$ (lower figure).

Action	Wilson	SW	D234c	
a (fm)	0.4	0.4	0.25	0.4
P	16.0(7)%	8(2)%	1.0(1)%	2.0(7)%
V	15.2(11)%	8(2)%	1.3(2)%	2.0(5)%
N	9.0(14)%	5(2)%	0.7(2)%	1.1(6)%
D	9.0(20)%	8(2)%	0.9(2)%	1.5(6)%

Table 4: Percentage change in hadron masses when the redundant coefficient r is changed from 1 to $2/3$. Scaling violation due to redundant terms at $r = 1$ is therefore expected to be roughly 3 times this.

which this is true gives us an indication of the size of the finite- a errors associated with the r terms, including the dominant correction, the clover term. Our results are summarized in Table 4. As expected, the variation with r is much smaller with D234c than with the SW or Wilson actions—the r terms are more highly corrected in the former. This “ r -test” suggests that r -dependent finite- a errors in the static masses are of order 3–4% at 0.25 fm, and 6% at 0.4 fm for D234c. Our discussion above shows that these errors could be almost entirely due to radiative corrections to the clover coupling C_F .

5 Conclusions

Our studies show that at tree level the mean-link tadpole-improved D234c action is accurate to about 10% in hadronic masses, even for lattice spacings as large as 0.4 fm and meson masses as large as $2/a \approx 1$ GeV. The meson dispersion relation ($c^2(\mathbf{p})$) is similarly accurate out to three-momenta of order $1.5/a \approx 750$ MeV. It is markedly superior to the SW action in the limit of large masses or momenta.

Our analysis indicates that the leading source of error in the hadron masses is the radiative correction, beyond tadpole-improved tree-level, to the clover coefficient C_F , which enters in the $\mathcal{O}(a)$ correction. Radiative corrections to other terms contribute probably less than 5% at $a = 0.4$ fm, while uncorrected a^4 errors are probably no more than a few percent. Thus the errors in the hadron spectrum in our 0.4 fm simulation can almost certainly be reduced to less than 10% by correcting C_F .

We estimated the size of the quantum corrections to C_F by tuning C_F until our hadron masses, computed with $a = 0.25$ fm and 0.4 fm, agreed with a quadratic $a = 0$ extrapolation of SCRI’s (tadpole-improved) SW data. We found that

the quantum corrections to C_F are of the right size to be perturbative. The perturbative prediction for C_F is not yet known, but we may compare with the clover coefficient in the SW action which, for Wilson’s (unimproved) gluon action, has the perturbative expansion $1+0.46\alpha_s$ after mean-link tadpole-improvement [4], which agrees with non-perturbative estimates to within a reasonably sized $\mathcal{O}(\alpha^2)$ error.

We also demonstrated how to nonperturbatively tune the leading $\mathcal{O}(a^2)$ correction, the C_3 term in the D234c action, by using restoration of rotational invariance. The correction to tadpole-improved tree-level is small, and is unimportant at the few percent level for $a \lesssim 0.4$ fm.

As expected, all errors were much smaller at $a = 0.25$ fm, and radiative corrections were much less important. With a properly tuned clover coefficient, C_F , D234c should give (static and kinetic) hadron masses accurate to within a couple of percent at this relatively large lattice spacing.

In the near future we will complete perturbative calculations of C_F and C_3 to compare with our nonperturbative tunings; nonperturbative tuning of C_F using PCAC restoration is another possibility [17]. We are also examining currents, form factors and other quantities with the D234c action. Perhaps most importantly, we are continuing our previous work on D234c for anisotropic lattices. This simulation technology is applicable to a range of phenomenological applications (hadronic spectra and decay constants, relativistic heavy-quark physics, high-momentum form factors), and we anticipate being able to obtain useful results with spatial lattice spacings in the range 0.2–0.4 fm.

References

- [1] M. Lüscher, S. Sint, R. Sommer, P. Weisz, H. Wittig and U. Wolff, [hep-lat/9608049](#).
- [2] B. Sheikoleslami and R. Wohlert, Nucl. Phys. **B259** (1985) 609.
- [3] G.P. Lepage and P.B. Mackenzie, Phys. Rev. **D48** (1993) 2250.
- [4] P. Lepage, [hep-lat/9707026](#), to appear in proceedings of “Lattice QCD on Parallel Computers” (Tsukuba, March 1997).
- [5] M. Alford, T.R. Klassen, G.P. Lepage, unpublished.
- [6] H. Trottier, Phys. Rev. **D55** (1997) 6844.
- [7] M. Alford, T.R. Klassen, G.P. Lepage, Nucl. Phys. **B** (Proc. Suppl), **47** (1996) 370.

- [8] M. Alford, T.R. Klassen, G.P. Lepage, Nucl. Phys. **B** (Proc. Suppl), **53** (1997) 861.
- [9] M. Alford, T.R. Klassen, G.P. Lepage, Nucl. Phys. B496 (1997) 377.
- [10] P. Lepage, **hep-lat/9607076**, to appear in proceedings of 1996 Schlading Winter School on Perturbative and non-Perturbative aspects of Quantum Field Theory.
- [11] P. Weisz and R. Wohlert, Nucl. Phys. **B236** (1984) 397, erratum **B247** (1984) 544
- [12] M. Lüscher and P. Weisz, Comm. Math. Phys. **97** (1985) 59, erratum **98** (1985) 433.
- [13] M. Alford, W. Dimm, G.P. Lepage, G. Hockney, P.B. Mackenzie, Phys. Lett. **B361** (1995) 87.
- [14] S. Collins, R. Edwards, U. Heller and J. Sloan, **hep-lat/9608021**.
- [15] T. R. Klassen, **hep-lat/9612005**, Proceedings of “Multi-Scale Phenomena and Their Simulation” Bielefeld, Germany, Oct 1996; F. Karsch (ed.)
- [16] P. Lepage and H. Trottier, in progress.
- [17] R. Edwards, U. Heller, T. Klassen **hep-lat/9710054**, to appear in proceedings of International Symposium on Lattice Field Theory, Edinburgh, 1997.
- [18] G.P. Lepage, L. Magnea, C. Nakhleh, U. Magnea and K. Hornbostel, Phys. Rev. **D46** (1992) 4052.

Appendix A Galilean Quarkonium

As explained in Section 3, Galilean quarkonium is a convenient fictitious state for the evaluation of ratios of lattice spacings. It is simulated using the conventional NRQCD quark action, without relativistic corrections:

$$G(\vec{x}, t + a) = \left(1 - a \frac{H_0}{2n}\right)^n \left(1 - a \frac{\delta H}{2}\right) U_{\vec{x}, t}^\dagger \left(1 - a \frac{\delta H}{2}\right) \left(1 - a \frac{H_0}{2n}\right)^n G(\vec{x}, t) \quad (\text{A.1})$$

where

$$H_0 = -\frac{\Delta^{(2)}}{2M} \quad (\text{A.2})$$

$$\delta H = -\frac{a}{4n} \frac{(\Delta^{(2)})^2}{4M^2} + \frac{a^2}{24M} \Delta^{(4)} \quad (\text{A.3})$$

For details of the lattice derivatives and fields, see [18].

For this paper, we needed the ratios of our two lattice spacings to each other, and to SCRI's $\beta = 7.4$ improved glue. We therefore performed Galilean charmonium simulations using all 3 types of glue, at a range of quark masses. The quark mass is characterized by the dimensionless quantity $M_{\text{kin}}/(P - S)$. M_{kin} is the kinetic mass of the quarkonium S -state, obtained from measurements of the energy of the S -state at rest and with the smallest momentum on the lattice. This quantity is 6.8 for charmonium, so our Galilean quarkonium states were mostly lighter than charmonium.

Our raw measurements of the $P - S$ splitting are given in table 5, and the resultant lattice spacing ratios are in table 6. We see that there is no dependence of the ratios on the quark mass within statistical errors, so that finite- a errors in our ratios are negligible.

Using a potential model for Galilean quarkonium, we find that the radius of the meson changes by 25 – 30% over this range of quark masses, so any uncorrected a^4 errors would double in size. Interestingly, the potential model gives similar sensitivities of $P - S$ splitting to quark mass to those measured on the lattice (last line of table 5).

β	$\frac{M_{\text{kin}}}{(P - S)}$			
	3.6	4.0	5.2	7.2
1.157	1.0846(33)	1.0545(60)	1.0124(70)	0.9667(80)
1.719	0.6863(35)	0.6729(50)	0.6317(80)	0.5990(51)
7.4 (SCRI)	0.6638(36)	0.6513(67)		
$\frac{d \ln(P - S)}{d M_{\text{kin}}/(P - S)}$	-0.06(1)	-0.05(1)	-0.04(2)	-0.02(1)

Table 5: Measurements of $a(P - S)$ for Galilean quarkonium at various kinetic quarkonium masses M_{kin} . Also (final line), sensitivity of $P - S$ to quark mass.

	$\frac{M_{\text{kin}}}{(P - S)}$			
	3.6	4.0	5.2	7.2
$\frac{a(\beta=1.157)}{a(\beta=1.719)}$	0.633(4)	0.638(6)	0.624(9)	0.620(8)
$\frac{a(\beta=1.157)}{a(\text{SCRI}, \beta=7.4)}$	0.612(4)	0.618(7)		
$\frac{a(\beta=1.719)}{a(\text{SCRI}, \beta=7.4)}$	0.967(7)	0.968(12)		

Table 6: Ratios of lattice spacings calculated using Galilean quarkonium, for a variety of quarkonium masses. Our glue ($\beta = 1.157, 1.719$) is mean-link tadpole-improved, SCRI's ($\beta = 7.4$) is plaquette tadpole-improved.

Appendix B Data Tables

β	lattice	am_{quark}	$a(P - S)$	a (fm)	a^{-1} (MeV)
1.157	$5^3 \times 10$	2.75	0.925(7)	0.400(4)	495(4)
1.719	8^4	1.80	0.578(10)	0.249(5)	790(10)

Table 7: Determination of lattice spacing from NRQCD calculation of spin-averaged charmonium $P - S$ splitting, $P - S = 458$ MeV. The lattice size and bare quark mass are given.

	D234c, $am_{\text{quark}} = 0.640$			SW, $am_{\text{quark}} = 0.594$		
	6^3	7^3	8^3	6^3	7^3	8^3
P	0.901(5)	0.917(6)	0.908(4)	0.939(6)	0.955(6)	0.945(4)
V	1.293(9)	1.312(12)	1.304(8)	1.332(8)	1.350(15)	1.333(8)
N	1.950(20)	1.955(15)	1.950(25)	1.960(40)	1.990(20)	1.958(30)
D	2.210(20)	2.210(20)	2.220(20)	2.250(20)	2.220(40)	2.230(25)

Table 8: Finite volume errors: Hadron masses in lattice units at $P/V \approx 0.7$, $a = 0.25$ fm, measured on $6^3 \times 20$, $7^3 \times 20$, and $8^3 \times 20$ lattices.

	$a = 0.25$ fm		$a = 0.40$ fm	
	D234c	SW	D234c	SW
quark	0.68	0.63	1.002	1.012
P/V	0.756(5)	0.762(6)	0.756(4)	0.763(3)
P	1.051(4)	1.085(4)	1.583(3)	1.497(7)
V	1.391(7)	1.424(7)	2.095(10)	1.960(10)
N	2.100(30)	2.150(24)	3.220(15)	3.000(20)
D	2.340(30)	2.370(20)	3.540(30)	3.260(20)

Table 9: Hadron masses in lattice units at $P/V \approx 0.76$. For $a = 0.40$ fm we used a $5^3 \times 18$ lattice, for $a = 0.25$ fm an $8^3 \times 20$. Physical spatial volume is the same.

	$a = 0.25$ fm		$a = 0.40$ fm	
	D234c	SW	D234c	SW
quark	0.640	0.594	0.943	0.970
P/V	0.694(3)	0.708(8)	0.703(4)	0.699(8)
P	0.902(3)	0.945(4)	1.371(3)	1.313(15)
V	1.300(4)	1.333(8)	1.950(10)	1.878(10)
N	1.925(25)	1.958(30)	2.960(20)	2.850(20)
D	2.170(20)	2.230(25)	3.380(20)	3.150(30)

Table 10: Hadron masses in lattice units at $P/V \approx 0.70$. For $a = 0.40$ fm we used a $5^3 \times 18$ lattice, for $a = 0.25$ fm an $8^3 \times 20$. Physical spatial volume is the same.

mom	$c^2(P)$		$c^2(V)$	
	$C_F = 1$	$C_F = 0.72$	$C_F = 1$	$C_F = 0.72$
001	0.95(2)	0.99(2)	1.05(5)	0.93(5)
100	0.94(4)	0.91(3)	1.08(3)	0.91(3)
101	0.94(4)	0.96(2)	1.16(4)	0.96(4)
002	0.90(2)	0.93(4)		
110	0.90(3)	0.89(5)	1.30(10)	0.92(6)
111	0.84(5)	0.98(9)		

Table 11: Sensitivity of c^2 to C_F : Hadron c^2 at $P/V = 0.76$, $5^2 \times 8 \times 18$, $a = 0.4$ fm lattice. The vector particle's c^2 shows definite sensitivity to a change in C_F (P/V kept constant)

mom	$c^2(P)$			$c^2(V)$		
	SW	$C_3 = 1$	$C_3 = 1.2$	SW	$C_3 = 1$	$C_3 = 1.2$
001	0.632(24)	0.96(2)	1.01(2)	0.421(16)	1.05(5)	1.23(5)
100	0.550(13)	0.93(3)	1.01(2)	0.342(13)	1.08(3)	1.23(6)
101	0.560(14)	0.93(3)	1.02(2)	0.355(14)	1.17(4)	1.32(6)
002	0.510(16)	0.90(2)	0.99(2)			
110	0.516(20)	0.90(3)	1.08(3)	0.345(13)	1.28(9)	
111	0.505(60)	0.84(5)	1.01(8)			

Table 12: Sensitivity of c^2 to C_3 , the coeff of $\Delta^{(3)}$. Tree-level value is $C_3 = 1$, SW has $C_3 = 0$. The table shows hadron c^2 at $P/V = 0.76$ on a $5^2 \times 8 \times 18$, $a = 0.4$ fm lattice. From the pseudoscalar c^2 , $C_3 = 1.2$ appears to be the results that would follow from non-perturbative tuning.

	$c^2(P)$		$c^2(V)$	
mom	SW	D234c	SW	D234c
100	0.79(3)	0.99(3)	0.68(6)	0.97(5)
110	0.76(4)	0.95(4)	0.65(3)	0.96(4)
111	0.84(9)	0.98(9)		
200	0.70(9)	0.94(8)		

Table 13: Hadron c^2 at $P/V = 0.76$, on an $8^3 \times 20$, $a = 0.25$ fm lattice.